Linear Programming: Duality cont'd.

Last time:

Given primal LP $\min c^T x$, $Ax \geq b$, $x \geq 0$; dual is $\max b^T y$, $A^T y \leq c$, $y \geq 0$

Theorem (weak duality): Let x, y be feasible solns. for primal & dual, then

$$c^T x \geq b^T y.$$

Let's try & get the dual LP for more general forms:

(P) $\min c^T x$

$\forall j = 1 \dots d \quad A_j x \geq b_j \quad x\, y_1 \dots y_d$

$\forall j = d+1 \dots m \quad A_j x = b_j \quad x\, y_{d+1} \dots y_m$

$\forall i = 1 \dots k \quad x_i \geq 0$

$\forall i = k+1 \dots n \quad x_i$ free

$$b^T y \leq \sum_{j=1}^{d} y_j (A_j x) + \sum_{j=d+1}^{m} y_j (A_j x) = \sum_{j=1}^{m} y_j A_j x = \sum_{j=1}^{m} \sum_{i=1}^{n} y_j A_{ji} x_i$$

$A_j x \geq b_j$, $j = 1 \dots d$

$A_j x = b_j$, $j = d+1 \dots m$

$y_j \geq 0$, $j = 1 \dots d$

$$= \sum_{i=1}^{n} x_i \sum_{j=1}^{m} A_{ji} y_j = \sum_{i=1}^{n} x_i (A^{iT} y)$$

$$\leq \sum_{i=1}^{n} c_i x_i \quad \text{———} \; (*)$$

$x_i \geq 0 \quad i = 1 \dots k$

$A^{iT} y \leq c_i \quad i = 1 \dots k$

$A^{iT} y = c_i \quad i = k+1 \dots n$

So now our dual LP is:

(D) $\max b^T y$

$$A^{iT} y \leq c_i \quad i = 1 \dots k$$

$$A^{iT} y = c_i \quad i = k+1 \dots n$$

$$y_j \geq 0 \quad j = 1 \dots d$$

$$y_j \text{ free} \quad j = d+1 \dots m$$

	In primal/dual		In dual/primal
thus:	non negative variables	$\Leftrightarrow$	inequality constraints
	free variables	$\Leftrightarrow$	equality constraints.

do A first

Complementary Slackness:

Consider ⊛. Suppose we are given x^*, y^* that are feasible (for primal & dual respectively). Further $c^T x^* = b^T y^*$. Clearly, x^*, y^* are optimal solutions.

Further, both inequalities in ⊛ must be equality, i.e.,

$\forall j = 1 \dots d$, ~~either~~ $\cancel{y_j (A_j x) = 0}$

either $y_j = 0$ or $A_j x = b_j$ — (I)

$\forall i = 1 \dots k$, either $x_i = 0$ or $A^{iT} y = c_i$ — (II)

The converse can also be seen to be true: given x^*, y^* that are feasible and satisfy (I), (II), ~~at~~ $c^T x^* = b^T y^*$, and hence these are optimal solutions for primal & dual.

Theorem (Complementary Slackness): Given LP (P) and dual (D), feasible solutions x & y are optimal if and only if they satisfy conditions (I) & (II).

A

do this before comp. slackness

Theorem (strong duality): Let x^*, y^* be optimal solutions for (P) & (D) respectively. Then $c^T x^* = b^T y^*$, i.e., if either primal or dual optimal solution exists, then the two must have the same value.

(w/o proof).

Cases:	Primal	Dual
	Optimal value exists	Optimal value exists
	Unbounded optimal	infeasible
	Infeasible	Unbounded optimal
	Infeasible	Infeasible

(why can't only one have an optimal solution?)

An application of duality: Max-Flow Min-cut.

Recall: given a directed graph w/ edge capacities, find maximum s-t flow / minimum s-t cut

c_e for a cut $X, V\setminus X$ s.t. $s \in X$, $t \notin X$

capacity of cut $= \sum_{\substack{e=(u,v) \\ u \in X,\, v \notin X}} c_e$

One direction was easy: max-flow $\leq$ min-cut, since given any ~~flo.~~ s-t flow f, any cut $X, V \setminus X$ s.t. $s \in X$, $t \notin X$

value of flow

$$= \sum_{\substack{e \text{ out of} \\ X}} f_e - \sum_{\substack{e \text{ into} \\ X}} f_e \leq \sum_{\substack{e \text{ out of} \\ X}} f_e \leq \sum_{\substack{e \text{ out of} \\ X}} c_e = \text{capacity of cut.}$$

Will get the other direction using strong duality.

LP for max-flow:

$$\max \sum_{e \text{ out of } s} x_e - \sum_{e \text{ into } s} x_e$$

$$\text{s.t.} \quad \forall v \neq s,t \quad \sum_{\substack{\cancel{e \text{ out of}} \\ e \text{ into } v}} x_e - \sum_{e \text{ out of } v} x_e = 0 \qquad \times \alpha_v$$

$$\forall e \qquad x_e \leq c_e \qquad \times \beta_e$$

$$\forall e \qquad x_e \geq 0$$

Let's write the dual for this:

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$$\min \sum_e \beta_e c_e$$

constraint matrix for primal:

incident on s or v

$$\begin{array}{c c}
 & |E| \\
|U|-2 & \left[\begin{array}{ccccc}
+1 & +1 & +1 & +1 & 0 \\
-1 & & \vdots & 0 & -1 \\
0 & -1 & -1 & \vdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 \\
\hline
1 & & & & \\
 & 1 & & & \\
 & & 1 & & \\
 & & & \ddots & \\
 & & & & 1
\end{array}\right] \\
|E| &
\end{array}$$

$$\underset{(-)}{u} \xrightarrow{e} \underset{(+)}{v}$$

constraints for dual:

$$\left.\begin{array}{llr}
\forall e=(u,v),\ (u,v)\notin\{s,t\} & \alpha_v - \alpha_u + \beta_e & \geq 0 \\
\forall e=(u,t) & -\alpha_u + \beta_e & \geq 0 \\
\forall e=(t,v) & \alpha_v + \beta_e & \geq 0 \\
\forall e=(s,v) & \alpha_v + \beta_e & \geq 1 \\
\forall e=(u,s) & -\alpha_u + \beta_e & \geq -1
\end{array}\right\} \quad \begin{array}{l} \alpha_v - \alpha_u + \beta_e \geq 0 \quad \forall e=(u,v) \\ \alpha_s = 1 \\ \alpha_t = 0 \end{array}$$

α free, $\beta \geq 0$

OR:

$$\min \sum_e \beta_e c_e$$

$\forall e=(u,v) \qquad \beta_e \geq \alpha_u - \alpha_v$

$\alpha_s = 1$

$\alpha_t = 0$

α free, $\beta \geq 0$.

to see correspondence b/w dual LP & cuts,

given cut $(X, V/X)$, set $\alpha_v = 1 \ \forall v \in X$, $\alpha_v = 0$ o.w.
$s \in X$, $t \notin X$

$$\beta_e = 1 \quad \text{for } e = (u,v),\ u \in X,\ v \notin X$$
$$= 0 \quad \text{o.w.}$$

the (α, β) is feasible for dual, value $= \sum_e c_e \beta_e$ = capacity of cut.

We will show that optimal solution to dual gives a cut of value equal to maximum flow, thus proving max-flow min-cut theorem.

Will use complementary slackness.

Let x^*, (α^*, β^*) be optimal primal & dual solutions (then x^* is a max s-t flow).

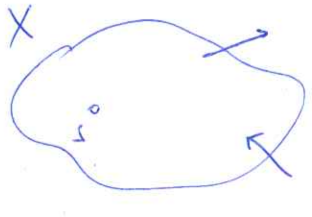
Note: by CS, $x_e^* > 0 \Rightarrow \beta_e^* = \alpha_u^* - \alpha_v^*$

$\beta_e^* > 0 \Rightarrow x_e^* = c_e$

now $\alpha_s^* = 1$. Let $X = \{v : \alpha_v^* \geq 1\}$. then $s \in X$, $t \notin X$

maximum flow $= \sum_{e \text{ out of } s} x_e^* - \sum_{e \text{ into } s} x_e^*$

$= \sum_{e \text{ out of } X} x_e^* - \sum_{e \text{ into } X} x_e^*$



or

Now consider an edge $e = (u,v)$ out of X.
then $\alpha_u^* \geq 1$, $\alpha_v^* < 1 \Rightarrow \beta_e^* > 0$
$\Rightarrow x_e^* = c_e$

for edge $e = (u,v)$ in to X. If $x_e^* > 0$,
$\alpha_u^* < 1$, $\alpha_v^* \geq 1 \Rightarrow \beta_e^* < 0$

hence for an edge e into X, $x_e^* = 0$

hence, maximum flow $= \sum_{e \text{ out of } X} c_e$, and this is the minimum cut.

Bipartite Matching:

Here is the LP corresponding to bipartite matching:

Primal: $\min \sum_e x_e$

$\forall v \quad \sum_{e \text{ incident on } v} x_e \leq 1$

$\forall e \quad x_e \geq 0$

Dual: $\max \sum_v y_v$

$\forall e = (u,v) \quad y_u + y_v \leq 1$

$\forall v \quad y_v \geq 0$

The dual corresponds to the vertex cover problem:

given a graph $G = (V, E)$, find a minimum size set of vertices X s.t. for every edge, at least one end-pt. is in X.

This is NP-hard in general graphs.

But can be solved in polynomial-time in bipartite graphs, since the constraint matrix is TU!

(hence we can solve the dual LP in polynomial time & obtain an integral optimal soln.)